Decompositions of Reflexive Modules

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Abstract

In a recent paper [11] we answered to the negative a question raised in the book by Eklof and Mekler [8, p. 455, Problem 12] under the set theoretical hypothesis of $\diamondsuit_{\aleph_1}$ which holds in many models of set theory. The Problem 12 in [8] reads as follows: *If A is a dual (abelian) group of infinite rank, is $A \cong A \oplus \mathbb{Z}$?* The set theoretic hypothesis we made is the axiom $\diamondsuit_{\aleph_1}$ which holds in particular in Gödel's universe as shown by R. Jensen, see [8]. Here we want to prove a stronger result under the special continuum hypothesis (CH).

The question in [8] relates to dual abelian groups. We want to find a particular example of a dual group, which will provide a negative answer to the question in [8]. In order to derive a stronger and also more general result we will concentrate on reflexive modules over countable principal ideal domains R. Following H. Bass [1] an R-module G is reflexive if the evaluation map $\sigma : G \longrightarrow G^{**}$ is an isomorphism. Here $G^* = \mathrm{Hom}\,(G, R)$ denotes the dual group of G. Guided by classical results the question about the existence of a reflexive R-module G of infinite rank with $G \not\cong G \oplus R$ is natural, see [8, p. 455]. We will use a theory of bilinear forms on free R-modules which strengthens our algebraic results in [11]. Moreover we want to apply a model theoretic combinatorial theorem from [14] which allows us to avoid the weak diamond principle. This has the great advantage that the used prediction principle is still similar to $\diamondsuit_{\aleph_1}$ but holds under CH. This will simplify algebraic argument for $G \not\cong G \oplus R$.

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1 Introduction

Let R be a countable principal ideal domain with $1 \neq 0$ and S a multiplicative closed subset of $R \setminus \{0\}$ containing 1. If $S = \{s_n : n \in \omega\}$, $s_0 = 1$ and $q_n = \prod_{i<n} s_i$, then

$$q_{n+1} = q_n s_n \text{ and } \bigcap_{n \in \omega} q_n R = 0. \tag{1.1}$$

The condition (1.1) requires that R is not a field, but then we may choose $S = R \setminus \{0\}$ or any other 'classical example'. The right ideals $q_n R$ form a basis of the S-topology on R which is Hausdorff by (1.1). The S-adic completion of R under the S-topology is the ring $\widehat{R}$ which has the size $2^{\aleph_0}$, see Göbel, May [10] for properties on $\widehat{R}$. Similar consideration carry over to (free) R-modules which we will use in the next sections. If F is a free R-module then the S-topology is generated by Fq_n $(n \in \omega)$ and if $\widehat{F}$ denotes the S-adic completion of F then $F \subset \widehat{F}$ and F is pure and dense in $\widehat{F}$. Recall that $F \subseteq_* \widehat{F}$ is pure (w.r.t. S) if and only if

$$\widehat{F}s \cap F \subseteq Fs \text{ for all } s \in S.$$

Also F is dense in $\widehat{F}$ if and only if $\widehat{F}/F$ is S-divisible (in the obvious sense). Note that an element e of an R-module G is pure, we write $e \in_* G$ if $eR \subseteq_* G$. If $X \leq G$, then $\langle X \rangle$ denotes the submodule generated by X and $\langle X \rangle_*$ denotes the submodule purely generated by X. The following observation is well-known and can be looked up in [10].

Observation 1.1 *If $0 \neq r_n \in R$ infinitely often $(n \in \omega)$, then we can find $\epsilon_n \in \{0, 1\}$ such that*

$$\sum_{n \in \omega} r_n q_n \epsilon_n \in \widehat{R} \setminus R.$$

If G is any R-module then $G^* = \mathrm{Hom}\,(G, R)$ denotes its dual module, and G is a dual module if $G \cong D^*$ for some R-module D. Particular dual modules are reflexive modules D introduced by Bass [1, p. 476]. We need the evaluation map

$$\sigma = \sigma_D : D \longrightarrow D^{**} \quad (d \longrightarrow \sigma(d))$$

where $\sigma(d) \in D^{**}$ is defined by evaluation

$$\sigma(d) : D^* \longrightarrow R \quad (\varphi \longrightarrow \varphi(d)).$$

The module D is *reflexive* if the evaluation map σ_D is an isomorphism. Obviously $D \cong (D^*)^*$ is a dual module if D is reflexive. However, there are very many $\aleph_1$-free

modules G with $G^* = 0$, see [2]. An R-module is $\aleph_1$-free if all countable submodules are free. Using the special continuum hypothesis CH the existence of many reflexive modules (Section 3) will follow from considerations of free modules with bilinear form in Section 2. Classical examples are due to Specker and Łos and it follows from Specker's theorem that reflexive modules must be $\aleph_1$-free, see Fuchs [9]. Hence it is not too surprising that we will deal with free R-modules first in Section 2. The bilinear form is needed to control their duals when passing from $\aleph_0$ to size $\aleph_1$ in Section 3.

In order to find reflexive groups G of cardinality $\aleph_1$ with $G \not\cong R \oplus G$ we must discard all possible monomorphisms

$$\varphi : G \hookrightarrow G \text{ with } G\varphi \oplus eR = G. \tag{1.2}$$

This will be established with the help of an $\aleph_1$-filtration

$$G = \bigcup_{\alpha \in \omega_1} G_\alpha \text{ of countable, pure, free submodules } G_\alpha$$

such that G_α is a summand with R-free complement of any G_β for $\alpha < \beta$ if α does not belong to a fixed stationary subset E of ω_1. Given φ as in (1.2) by a back and forth argument there is a cub $C \subseteq \omega_1$ such that $G_\alpha\varphi \subseteq G_\alpha$ and $e \in G_\alpha$ for all $\alpha \in C$.

If $\alpha \in E \cap C$ then

$$G_\alpha\varphi \oplus eR = G_\alpha. \tag{1.3}$$

And if $\varphi \restriction G_\alpha$ is predicted by a function φ_α as under the assumption of the diamond principle $\diamondsuit_{\aleph_1}$, then we construct $G_{\alpha+1}$ by a Step-Lemma from G_α and φ_α such that φ_α does not extend to $G_{\alpha+1}$. Moreover $G_{\alpha+1}$ is the S-adic closure of G_α in G by the summand property mentioned above. Hence φ coincides with φ_α on G_α and maps $G_{\alpha+1}$ into itself, a contradiction. However assuming CH only weaker prediction principles like weak diamond $\Phi_{\aleph_1}$ are available, see Devlin and Shelah [3] or Eklof and Mekler [8, p. 143, Lemma 1.7]. If we discard φ_α in the construction as before, then $\varphi \restriction G_\alpha$ might extend because it does not entirely agree with φ_α. In this case φ will show up at some $\alpha < \beta \in E \cap C$ and we have a new chance to discard $\varphi \restriction G_{\alpha+1}$ at level β - provided we know $\varphi \restriction G_{\alpha+1}$ when constructing $G_{\beta+1}$.

This time we need a stronger algebraic algebraic Step-Lemma: Note that $G_{\alpha+1} \oplus F_\beta = G_\beta$. Now we have a partial map $\varphi_\beta := \varphi \restriction G_{\alpha+1}$ with domain $\operatorname{Dom}\varphi_\beta = G_{\alpha+1}$ a summand of G_β and the splitting property (1.3) for the new map φ_β:

$$(G_{\alpha+1}\varphi_\beta \oplus eR) \oplus F_\beta = G_\beta. \tag{1.4}$$

In order to proceed we must discard these partial maps and indeed we are able to prove a generalized Step-Lemma taking care of partial maps (1.4) in Section 2. Moreover, a

counting argument also shows that a list of such partial maps

$$\{\varphi_\beta : \ \beta \in E\}$$

exists which predict any given map $\varphi : G \longrightarrow G$ such that the following holds.

Lemma 1.2 *[14] Let $G = \bigcup_{\alpha \in \omega_1} G_\alpha$ be an $\aleph_1$-filtration of G and $E \subseteq \omega_1$ be a stationary subset of ω_1. Then there is a list of predicting partial maps*

$$\{\varphi_\alpha : G_\alpha \longrightarrow G_\alpha : \ \alpha \in E\}$$

such that for any countable subset A of G there is an ordinal $\beta \in E$ (in fact an unbounded set of such ordinals) with

$$\varphi \restriction A = \varphi_\beta \restriction (G_\beta \cap A).$$

For a suitable $A = G_{\alpha+1}$ and $\varphi \restriction A$ we choose $\beta \in E$ by Lemma 1.2 with $\varphi \restriction G_{\alpha+1} = \varphi_\beta \restriction G_{\alpha+1}$ and $G_\beta = G_{\alpha+1} \oplus F_\beta$. Then we construct $G_{\beta+1}$ from G_β and the given $\varphi_\beta \restriction G_{\alpha+1}$ such that $\varphi_\beta \restriction G_{\alpha+1}$ does not extend to $G_{\beta+1}$. This contradiction will provide the

Main Theorem 1.3 *(ZFC + CH) If R is a countable domain but not a field, then there is a family of $2^{\aleph_1}$ pair-wise non-isomorphic reflexive R-modules G of cardinality $\aleph_1$ such that $G \not\cong R \oplus G$.*

We also would like to draw attention to a slight modification of the proof of the Main Theorem 1.3. In addition we may assume that G in the Main Theorem 1.3 is essentially indecomposable, that any decomposition into two summands has one summand free of finite rank. This follows from a split realization result Corollary 4.3 with $\operatorname{End} G \cong A \oplus \operatorname{Fin}(G)$ where A is any R-algebra which is free of countable rank. Recall that $\operatorname{Fin}(G)$ is the ideal of all endomorphisms of G of finite rank.

The formal proof for the prediction Lemma 1.2 is not complicated and uses repeatedly often the weak diamond prediction $\Phi_{\aleph_1}$. It is also clear from what we said that the underlying module theory is not essential for proving Lemma 1.2 and that it should be possible to replace modules by many other categories like non-commutative groups, fields or Boolean algebras. In order to cover all these possibilities, the prediction principle is formulated in terms of model theory and will appear in this setting in a forth coming book by Shelah [14, Chapter IX, Claim 1.5].

We close this introduction with some historical remarks. Using a theorem of Łoś (see Fuchs [9]) on slender groups, the first 'large' reflexive abelian groups are free groups or

(cartesian) products of $\mathbb{Z}$ - assuming for a moment that all cardinals under consideration are $< \aleph_m$, the first measurable cardinal. Also the members of the class of groups generated by $\mathbb{Z}$ and taking direct sums and products alternatively are reflexive, called Reid groups. Using a generalized 'Chase Lemma', which controls homomorphisms from products of modules into direct sums of modules, Dugas and Zimmermann-Huisgen [5] showed that the class of Reid groups is 'really large'. Nevertheless there are more reflexive groups - Eda and Otha [7] applied their 'theory of continuous functions on 0-dimensional topological spaces' to find reflexive groups not Reid-groups. As a by-product we also get dual groups which are not reflexive, see also [8]. All these groups G have the property that they are either free of finite rank or

$$\mathbb{Z} \oplus G \cong G. \tag{1.5}$$

As indicated in the abstract, we applied $\diamondsuit_{\aleph_1}$ to find examples G of size $\aleph_1$, where (1.5) is violated. The obvious question to replace $\diamondsuit_{\aleph_1}$ by CH was the main goal of this paper. The question whether the Main Theorem 1.3 holds in any model of ZFC remains open. On the other hand we are able to show that the conclusion of the Main Theorem 1.3 also follows in models of ZFC and Martin's axiom MA, see [12]. Hence CH is not necessary to derive the existence of these reflexive modules.

2 Free modules with bilinear form and partial dual maps

Definition 2.1 *Let $(\Phi, \mathfrak{F}_0, \mathfrak{F}_1)$ be a triple of a bilinear map $\Phi : F_0 \oplus F_1 \longrightarrow R$ for some countable, free R-modules F_i of infinite rank, $\mathfrak{F}_i \subseteq F_i^*$ $(i = 0, 1)$ families of dual maps subject to the following conditions*

(i) *Φ is not degenerated. This is to say if $\Phi(e, \ ) \in F_1^*$ or $\Phi(\ , f) \in F_0^*$ is the trivial map then $e = 0$ or $f = 0$, respectively.*

(ii) *Φ preserves purity, that is $\Phi(e, \ ) \in_* F_1^*$ if $e \in_* F_0$ and dually $\Phi(\ , f) \in_* F_0^*$ if $f \in_* F_1$.*

(iii) *$\mathfrak{F}_i$ is a countable, non-empty family of homomorphisms $\varphi : F_\varphi \longrightarrow R$ such that $\operatorname{Dom}\varphi = F_\varphi \subseteq_* F_i$. The set $\operatorname{Dom}\mathfrak{F}_i = \{F_\varphi : \varphi \in \mathfrak{F}_i\}$ is well-ordered by inclusion, for $i \in \{0, 1\}$.*

(iv) *For any $0 \neq x \in F_1$ and any finite subset $E \subset \mathfrak{F}_0$ we have $\ker E \not\subseteq \ker \Phi(\ , x)$, and dually for any $0 \neq y \in F_0$ and any finite subset $E \subset \mathfrak{F}_1$, we have $\ker E \not\subseteq \ker \Phi(y, \ )$.*

Here, and in the subsequent parts we use the following

Notation 2.2 *(i)* $\mathfrak{F}$ *is the collection of all triples* $(\Phi, \mathfrak{F}_0, \mathfrak{F}_1)$ *as in Definition 2.1.*

(ii) If $E \leq \operatorname{Hom}(G, H)$ *then* $\ker E = \bigcap_{\varphi \in E} \ker \varphi$.

(iii) Similarly $\ker \Phi(\ , E) = \bigcap_{e \in E} \ker \Phi(\ , e)$ *and dually.*

Next we define a partial order on $\mathfrak{F}$.

Definition 2.3 *(i)* $(\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \subseteq (\Phi', \mathfrak{F}'_0, \mathfrak{F}'_1) \Longleftrightarrow$

(ii) $\begin{cases} (a)\ \ \Phi \subseteq \Phi' \textit{ and } \operatorname{Dom} \Phi \subseteq_* \operatorname{Dom} \Phi' \textit{ is a pure submodule.} \\ (b)\ \ \textit{If } \varphi \in \mathfrak{F}_i \textit{ then there is a unique } \varphi' \in \mathfrak{F}'_i \textit{ such that } \varphi \subseteq \varphi' \textit{ and} \\ \operatorname{Dom} \varphi \subseteq_* \operatorname{Dom} \varphi'. \end{cases}$

We will construct the reflexive modules of size $\aleph_1$ by using an order preserving continuous map and let

$$p : {}^{\omega_1 >}2 \longrightarrow \mathfrak{F} \ \ (\eta \longrightarrow p_\eta) \tag{2.1}$$

from the tree $\mathbf{T}$ of all branches of length $\lg(\eta) = \alpha$

$$\eta : \alpha \longrightarrow 2 = \{0, 1\} \text{ for all } \alpha < \omega_1.$$

The order on $\mathbf{T}$ is defined naturally by extensions, i.e. if $\eta, \eta' \in \mathbf{T}$, then $\eta \leq \eta'$ if and only if $\eta \subseteq \eta'$ as maps. Hence $\lg(\eta) \leq \lg(\eta')$ and $\eta' \restriction \lg(\eta) = \eta$, and we will require that $p_\eta \subseteq p_{\eta'}$ by the ordering of $\mathfrak{F}$ as defined in Definition 2.3. If $\eta \in {}^{\omega_1}2$, then $p_{\eta \restriction \alpha}$ $(\alpha \in \omega_1)$ is linearly ordered and the triple

$$p_\eta = \bigcup_{\alpha \in \omega_1} p_{\eta \restriction \alpha} \tag{2.2}$$

is well-defined. In details we have $\Phi_\eta : F_{0\eta} \oplus F_{1\eta} \longrightarrow R$ is defined by continuity such that $\Phi_\eta = \bigcup_{\alpha \in \omega_1} \Phi_{\eta \restriction \alpha}$ and $F_{i\eta} = \bigcup_{\alpha \in \omega_1} F_{i\eta \restriction \alpha}$. The bilinear forms $\Phi_\eta : F_{0\eta} \oplus F_{1\eta} \longrightarrow R$ will be our candidates for modules G as in the Main Theorem 1.3. First we will show that $\mathfrak{F} \neq \emptyset$ and the arguments will be refined for Lemma 2.5.

Lemma 2.4 *The partially ordered set* $\mathfrak{F}$ *is non-empty.*

Proof. Choose $F_0 = F_1 = \bigoplus_{n\in\omega} e_n R$ and extend

$$\Phi(e_i, e_j) = \delta_{ij} = \begin{cases} 0 & if \quad i \neq j \\ 1 & if \quad i = j \end{cases}$$

linearly to get a bilinear map $\Phi : F_0 \oplus F_1 \longrightarrow R$. The map Φ satisfies Definition 2.1 for $\mathfrak{F}_0 = \mathfrak{F}_1 = \emptyset$. Next we want find $\mathfrak{F}_0 = \{\varphi_0\}$ and $\mathfrak{F}_1 = \{\varphi_1\}$. If φ_i $(i \in 2)$ satisfy Definition 2.1 (iv) and $\operatorname{Dom}\varphi_i = F_i$ then obviously

$$(\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}.$$

We will work for $\varphi = \varphi_0 : F_0 \longrightarrow R$ and enumerate $F_1 \setminus \{0\} = \{x_i : \ i \in \omega\}$. If $\varphi_i = \Phi(\ , x_i)$ then $\varphi_i \neq 0$ by (i) and for (iv) we must show that

$$\ker\varphi \not\subseteq \ker\varphi_i \text{ for all } i \in \omega \tag{2.3}$$

Write $F_0 = \bigoplus_{i\in\omega} L_i$ such that each $L_i = e_i R \oplus e_i' R$ is free of rank 2 and let $L_i' = \bigoplus_{j\neq i} L_j$ be a complement of L_i . If $L_i\varphi_i = 0$ then we use $\varphi_i \neq 0$ to find some $0 \neq y \in_* L_i'$ such that $y\varphi_i \neq 0$. Choose a new complement of L_i' and rename it $L_i = (e_i + y)R \oplus e_i' R$.

Hence $L_i\varphi_i \neq 0$ and there is a pure element $y_i \in L_i$ with $y_i\varphi_i \neq 0$. We found an independent family $\{y_i : i \in \omega\}$ with $F_0 = \bigoplus_{i\in\omega} y_i R \oplus C$ for some $0 \neq C \subseteq F_0$. Choose $\varphi \in \operatorname{Hom}(F_0, R)$ such that $y_i\varphi = 0$ for all $i \in \omega$ and $\varphi \restriction C \neq 0$. Hence

$$y_i \in \ker\varphi \setminus \ker\varphi_i \text{ for all } i \in \omega$$

and (2.3) holds. Hence Definition 2.1 holds and $\mathfrak{F}_1$ can be chosen dually. $\square$

The crucial step in proving the next result is again verification of Definition 2.1 (iv), this time for $\Phi'(\ , x)$. The proof is similar to the last one, hence we can be less explicit and just refine the old arguments.

Lemma 2.5 *Let $p = (\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$ be with $\operatorname{Dom}\Phi = F_0 \oplus F_1$ and $\varphi \in L^*$ for some pure submodule L of finite rank in F_0. Then we find $p \subseteq p' = (\Phi', \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$ with $\operatorname{Dom}\Phi' = F_0 \oplus F_1'$, $F_1' = F_1 \oplus xR$ and $\varphi \subseteq \Phi'(\ , x)$.*

Proof. First we want to extend φ to $\varphi' : F_0 \longrightarrow R$ such that

$$\ker E \not\subseteq \ker\varphi' \text{ for all finite } E \leq \mathfrak{F}_0. \tag{2.4}$$

Enumerate all $\ker E$ for $E \leq \mathfrak{F}_0$ finite by $\{K_i : \ 0 \neq i \in \omega\}$. Hence $K_i = \bigcap_{\varphi\in E_i} \ker\varphi$ is a pure submodule of F_0 with F_0/K_i free of rank $\leq |E_i|$. As in the proof of (2.5) we can

choose inductively $\bigoplus_{i\in\omega} L_i = F_0$ and $0 \neq y_i \in_* L_i \cap K_i$ for $i > 0$ and $L_0 = L$. For some $C \subseteq F_0$ we have $F_0 = \bigoplus_{i\in\omega} y_i R \oplus L \oplus C$. Now we extend $\varphi \in L^*$ such that $y_i\varphi' = 1$ and $\varphi \restriction C = 0$. Hence

$$y_i \in K_i \setminus \ker \varphi' \text{ for all } \ 0 \neq i \in \omega$$

and (2.4) holds. Finally we extend Φ to $\Phi' : F_0 \oplus (F_1 \oplus xR) \longrightarrow R$ by taking $\Phi'(\ ,x) = \varphi'$. Condition (2.4) carries over to

$$\ker E \not\subseteq \ker \Phi(\ ,x) \text{ for all finite } E \leq \mathfrak{F}_0.$$

From this it is immediate that $\ker E \not\subseteq \ker \Phi(\ ,y)$ for all $0 \neq y \in F_1'$ and finite sets $E \leq \mathfrak{F}_0$. Hence Definition 2.1(iv) holds, $p \leq (\Phi', \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$ and $\varphi \subseteq \Phi(\ ,x)$. $\square$

Next we move elements from $\operatorname{Hom}(F_\varphi, R)$ for some $\varphi \in \mathfrak{F}_i$ to $\mathfrak{F}_i$.

Lemma 2.6 *Let $p = (\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$ be with $\operatorname{Dom} \Phi = F_0 \oplus F_1$. If $\psi \in \operatorname{Hom}(F_\varphi, R)$ for some $\varphi \in \mathfrak{F}_0$ such that*

$$\ker \psi \cap \ker E \not\subseteq \ker \Phi(\ ,x) \textit{ for all finite } E \leq \mathfrak{F}_0 \textit{ and } x \in F_1,$$

then

$$p \leq p' = (\Phi, \mathfrak{F}_0', \mathfrak{F}_1) \in \mathfrak{F} \textit{ for } \mathfrak{F}_0' = \mathfrak{F}_0 \cup \{\psi\}$$

Proof. We only have to check Definition 2.1(iv) for finite subsets of $\mathfrak{F}_0' = \mathfrak{F}_0 \cup \{\psi\}$. But this follows by hypothesis on ψ.

The proof of the following observation is obvious.

Observation 2.7 *If $p_n = (\Phi_n, \mathfrak{F}_{0n}, \mathfrak{F}_{1n})$ $(n \in \omega)$ is an ascending chain of elements $p_n \in \mathfrak{F}$ and elements in $\mathfrak{F}_i$ are unions of extensions in $\mathfrak{F}_{in} \subseteq \mathfrak{F}_{in+1}$ $(n \in \omega)$, then $p = (\bigcup_{n\in\omega} \Phi_n, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$.*

Definition 2.8 *If $(\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$ with $\operatorname{Dom} \Phi = F_0 \oplus F_1$ then $\varphi \in F_0^*$ is essential for Φ if for any finite rank summand L of F_0 and any finite subset E of F_1 there is $g \in F_0 \setminus L$ with $g\varphi \neq 0$ and $\Phi(g,e) = 0$ for all $e \in E$.*

The notion '$\varphi \in F_1^*$ is essential for Φ ' is dual.

If $g\varphi = 0 = \Phi(g,e)$ for $e \in E$ and some $E \leq \mathfrak{F}_1$ then $g \in \langle \Phi(\ ,e) : e \in E \rangle \subseteq F_0^*$ by induction on $|E|$. Hence $\varphi \in F_0^*$ is essential for Φ is equivalent to say that φ is not in $\Phi(\ ,F_1)$ modulo summands of finite rank. This leads to the following

Observation 2.9 *Let* $(\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$ *with* $\operatorname{Dom}\Phi = F_0 \oplus F_1$. *If* $E \leq F_1$ *is a finite subset and* $\varphi \in F_0^*$ *inessential for* Φ *with* $\ker\varphi \subseteq \ker E$ *i.e.*

$$(\Phi(x,e) = 0 \textit{ for all } \ e \in E) \Longrightarrow x\varphi = 0,$$

then there is $e_0 \in \langle E \rangle \subseteq F_1$ *such that* $\varphi = \Phi(\ ,e_0)$.

Proof. By induction on $|E|$.

First Killing-Lemma 2.10 *Suppose* $\varphi \in F_0^*$ *is essential for* Φ *with* $\operatorname{Dom}\Phi = F_0 \oplus F_1$ *and* $p = (\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$. *Then we find* $p \leq p' \in \mathfrak{F}$ *with* $p' = (\Phi', \mathfrak{F}_0', \mathfrak{F}_1')$, $\operatorname{Dom}\Phi' = F_0' \oplus F_1$ *and* $F_0' = \langle F_0, y \rangle \subseteq_* \widehat{F}_0$ *for some* $y \in \widehat{F}_0$ *such that* φ *does not extend to* $\varphi' : F_0' \longrightarrow R$.

Remark A dual lemma holds for $\varphi \in F_1^*$.

Proof. Let $F_0 = \bigoplus_{i \in \omega} e_i R$ and $F_1 = \bigoplus_{n \in \omega} f_n R$. First we apply that φ is essential. It is easy to find inductively elements $g_n \in F_0 \setminus F^n$ with $F^n = \langle e_i, g_i : i < n \rangle_* \subseteq F_0$ such that the following holds

(i) $\Phi(g_n, f_i) = 0$ for all $i < n$.

(ii) $\bigoplus_{i<n} g_i R$ is a direct summand - also $\bigoplus_{i \in \omega} g_i R$ is a summand of G.

(iii) $g_n \varphi \neq 0$ for all $n \in \omega$.

Decompose ω into a disjoint union of infinite subsets S_i $(i \in \omega)$ and let $\{E_i : 0 \neq i \in \omega\}$ be an enumeration of all finite subsets of $\mathfrak{F}_1$ and write $K_i = \ker E_i$ for all $i > 0$. In order to get $K_i \not\subseteq \ker\Phi(x,\ )$ for all $i > 0, x \in F_0' \setminus F$, we choose for each $n \in S_i$ an element $k_n \in K_i$ such that $\Phi(g_n, k_n) \neq 0$. This is possible as F_1/K_i is free of finite rank $\leq |E_i|$, hence K_i is 'quite large'.

Now we use $g_n\varphi \neq 0$ $(n \in \omega)$ and apply Observation 1.1 (for S_0) and choose a sequence $\epsilon_n \in \{0,1\} (n \in \omega)$ and suitable $q_n \in S$ as in (1.1) such that

$$r = \sum_{n \in \omega} (g_n\varphi) q_n \epsilon_n \in \widehat{R} \setminus R \tag{2.5}$$

$$\text{If } n \in S_i, \text{ then } \sum_{j=0}^{n} \Phi(g_j, k_i) q_j \epsilon_j \not\equiv 0 \mod q_{n+1} \tag{2.6}$$

If $s \in \omega$ and $\epsilon_s = 1$, then let

$$y_s = \sum_{s \leq n \in \omega} (q_s)^{-1} q_n g_n \epsilon_n \in \widehat{F}_0$$

and consider the R-module $F_0' = \langle F_0, y_0 \rangle_* \subset \widehat{F}_0$, which can be generated by

$$F_0' = \langle F_0, y_s R: \ s \in \omega, \epsilon_s = 1 \rangle.$$

Note that F_0' is a countable R-module. It is easy to see that F_0' is free - either apply Pontryagin's theorem (Fuchs [9, p. 93]) or determine a free basis. The bilinear form $\Phi : F_0 \oplus F_1 \longrightarrow R$ extends uniquely to $\Phi' : F_0' \oplus F_1 \longrightarrow \widehat{R}$ by continuity and density. We want to show that

$$(\Phi', \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}. \tag{2.7}$$

First we claim that $\operatorname{Im} \Phi' \subseteq R$. By (i) we have

$$\Phi'(y_0, f_j) = \Phi'(\sum_{n \in \omega} \epsilon_n q_n g_n, f_j) = \sum_{n \in \omega} \Phi(g_n, f_j) q_n \epsilon_n = \sum_{n < j} \Phi(g_n, f_j) q_n \epsilon_n \in R$$

and $\operatorname{Im} \Phi' \subseteq R$ follows.

We also must check (iv) from Definition 2.1 for the new elements $y \in F_0' \setminus F_0$. It is enough to consider

$$\ker E \not\subseteq \ker \Phi(y_s, \ ) \text{ for all } s \in \omega \text{ with } \epsilon_s = 1.$$

By definitions and enumerations this is equivalent to say that

$$K_i \not\subseteq \ker \Phi(y_s, \ ) \text{ for each } i > 0.$$

If $n \in S_i$ and $n > s$ by (2.6) we have that

$$\Phi(y_s, k_i) = \sum_{j \in \omega} \Phi(g_j, k_i) q_j \epsilon_j \equiv \sum_{j=0}^{n} \Phi(g_j, k_i) q_j \epsilon_j \not\equiv 0 \mod q_{n+1}$$

Hence $k_i \notin \ker \Phi(y_s, \ )$ but $k_i \in K_i$ and (2.7) follows. Finally we must show that $\varphi \in F_0^*$ does not extend to $(F_0')^*$. By continuity $\varphi : F_0 \longrightarrow R$ extends uniquely to $\varphi' : F_0' \longrightarrow \widehat{R}$. However

$$y_0 \varphi = (\sum_{n \in \omega} g_n q_n \epsilon_n) \varphi' = \sum_{n \in \omega} (g_n \varphi) q_n \epsilon_n = r \in \widehat{R} \setminus R$$

by (2.5), hence φ does not extend to $F_0' \longrightarrow R$. $\square$

Second Killing-Lemma 2.11 *Let $p = (\Phi, \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$ and $\eta : F_\varphi \longrightarrow F_\varphi$ be some monomorphism with $F_\varphi = x_0 R \oplus F_\varphi \eta$ for some $\varphi \in \mathfrak{F}_0$ with $x_0 \in F_\varphi = \ker \varphi$. Then there is $p \le p' = (\Phi', \mathfrak{F}'_0, \mathfrak{F}'_1) \in \mathfrak{F}$ with $\mathrm{Dom}\, \Phi' = F'_0 \oplus F'_1, \varphi \subseteq \varphi' \in \mathfrak{F}'_0$ such that η does not extend to a monomorphism*

$$\eta'' : F_{\varphi'} \longrightarrow F_{\varphi'} \subset_* F''_0 \text{ with } F''_{\varphi'} = x_0 R \oplus F''_{\varphi'} \eta''$$

where $p' \le p'' := (\Phi'', \mathfrak{F}''_0, \mathfrak{F}''_1)$ and $\mathrm{Dom}\, \Phi'' = F''{}_0 \oplus F''_1$.

Proof. In order to satisfy Definition 2.1 (iv) for the new $p' \in \mathfrak{F}$ we argue similar to the First Killing Lemma 2.10. Let $\omega = \bigcup\limits_{i \in \omega} S_i$ be a decomposition into infinite subsets S_i and $\{K_i : 0 \ne i \in \omega\}$ be all kernels $K_i = \ker E_i$ for an enumeration of finite subsets E_i of $\mathfrak{F}_1$. The set S_0 will be used for killing η and the S_i $(i > 0)$ are in charge of K_i and (iv) above. Extending $\Phi \subset \Phi'$ we must ensure $\mathrm{Im}\, \Phi' \subseteq R$. Hence we construct an increasing sequence $s_n \in \omega$ $(n \in \omega)$ and pose more conditions on s_n later on. If $F' = \mathrm{Dom}\, \varphi$ then $F' = x_0 R \oplus F' \eta$ and if $x_i = x_0 \eta^i$ we get

$$F' = \bigoplus_{i<n} x_i R \oplus F' \eta^n \text{ for all } n \in \omega. \tag{2.8}$$

Note that $\bigoplus\limits_{i<n} x_i R$ is pure of finite rank, hence $F_0 = \bigoplus\limits_{i<n} x_i R \oplus C_n$ for some $C_n \subseteq F_0$. Now we construct $T_n = \bigoplus\limits_{s_n \le i < s_{n+1}} e_i R$ from $F_0 = \bigoplus\limits_{i \in \omega} e_i R$ and refine an argument from Göbel, Shelah [11]. Obviously $F_0 = \bigoplus\limits_{n \in \omega} T_n$ and let $e^n \in \mathrm{Hom}\, (F_0, R)$ $(n \in \omega)$ be defined by

$$e_i e^n = \delta_{i,n} = \begin{cases} 0 & if \quad i \ne n \\ 1 & if \quad i = n \end{cases}$$

for all $i \in \omega$. If $i < n$ then $\pi_i : F_0 \longrightarrow R$ denotes the projection modulo $\bigoplus\limits_{i \ne j < n} x_j R \oplus C_n$, moreover let $F_1 = \bigoplus\limits_{j \in \omega} f_j R$ be as before. We now seek for elements $w_n \in F_0$ $(n \in \omega)$ subject to the following four conditions

(i) $0 \ne w_n \in T_n$ and $w_n \eta \in T_n$.

(ii) $\Phi(w_n, f_n) = w_n e^k = w_n \eta e^k = 0$ for all $k < s_n$

(iii) If $n \in S_0$, then let π^*_n be the projection π_i with i maximal such that

$$w_n \pi_i = 0 \ne w_n \eta \pi_i, \;\; s_n \le i < s_{n+1}$$

(iv) If $n \in S_i, i > 0$, there is $y_n \in K_i$ such that $\Phi(w_n, y_n) \neq 0$.

Suppose $s_0, \ldots, s_n, w_0 \ldots, w_{n-1}$ are constructed and we want to choose w_n, s_{n+1}. Then pick $s_n < s_{n+1}$ such that

$$\{x_{s_n}, \ldots, x_{4s_n+1}\} \subseteq \bigoplus_{i \leq s_{n+1}} e_i R.$$

We want to choose $w_n = \sum\limits_{i=s_n}^{4s_n} x_i a_i^n$ for some $a_i^n \in R$. If $w_n e^k = 0$ for $k < s_n$, then $w_n \in T_n$. Moreover

$$w_n \eta = \sum_{i=s_n}^{4s_n} x_i a_i^n \eta = \sum_{i=s_n}^{4s_n} x_{i+1} a_i^n = \sum_{i=s_n+1}^{4s_n+1} x_i a_{i-1}^n$$

and $x_n \eta \in T$ because $x_{4s_n+1} \in \bigoplus\limits_{i \leq s_{n+1}} e_i R$ as well.

Hence (i) follows provided $w_n \neq 0$ is generated by those $x_i's$. The conditions (ii) can be viewed as a system of $3s_n$ homogeneous linear equations in $4s_n + 1 - s_n = 3s_n + 1$ unknowns $a_i^n \in R$. We find a non-trivial solution $w_n \neq 0$ by linear algebra. Hence (i) and (ii) hold. Condition (iii) follows by hypothesis on $\mathfrak{F}_1$ for E_i and $K_i = \ker E_i$. Condition (iii) finally follows by the action of η on $w_n \neq 0$ and the maximality of i with $s_n < i \leq 4s_n$ and $w_n \pi_i = 0$ for $\pi_n^* = \pi_i$. Hence $(i), \ldots, (iv)$ follow.

As in the proof of the First Killing Lemma 2.10 inductively we choose a strictly increasing sequence $m_j \in \omega$ $(j \in \omega)$. If m_j is defined up to $j \leq n$ we must choose m_{n+1} large enough such that

$$\sum_{j \leq n} q_{m_j} \Phi(w_j, y_n) \not\equiv 0 \mod q_{m_{n+1}}.$$

This needs inductively the hypothesis that $\sum\limits_{j \leq n} q_{m_j} \Phi(w_j, y_n) \neq 0$. If $n+1 \in S_i$, then $\Phi(w_{n+1}, y_{n+1}) \neq 0$ by (iv) and we may assume

$$\sum_{j \leq n+1} p_j^m \ \Phi(w_j, y_{n+1}) \neq 0$$

hence the inductive hypothesis follows and we can proceed. By Observation 1.1 and (iii) we also find $\epsilon_j \in \{0, 1\}$ $(j \in S_0)$ such that

$$\sum_{j \in S_0} (w_j \eta) \pi_j^* q_{m_j} \epsilon_j \in \widehat{R} \setminus R. \tag{2.9}$$

Now we are ready to extend Φ. Choose new elements

$$z_k = \sum_{j\geq k} w_j q_{m_j} (q_{m_k})^{-1} \in \widehat{F}_0$$

for all $k \in \omega$. Hence the submodule $F_0' = \langle F_0, z_0 \rangle_* \subseteq \widehat{F}_0$ purely generated by adding $z = z_0$ is generated by

$$F_0' = \langle F_0, z_k : k \in \omega \rangle.$$

Again we see that F_0' is a countable, free R-module. The map $\Phi : F_0 \oplus F_1 \longrightarrow R$ by continuity extends uniquely to

$$\Phi' : F_0' \oplus F_1 \longrightarrow \widehat{R}.$$

Recall that F_0'/F_0 is S-divisible, hence F_0 is S-dense in F_0' in the S-adic topology. First we must show that $\operatorname{Im} \Phi' \subseteq R$. We apply (ii) and continuity to see that

$$\Phi'(z, f_k) = \Phi'(\sum_{n\in\omega} q_{m_n} w_n, f_k) = \sum_{n\in\omega} q_{m_n} \Phi(w_n, f_k) = \sum_{n\leq k} q_{m_n} \Phi(w_n, f_k) \in R,$$

hence $\Phi' : F_0' \oplus F_1 \longrightarrow R$.

We also must show Definition 2.1 (iv) for the new elements $z_t \in F_0'$. We have $K_i = \ker E_i$ and S_i is unbounded. Hence we find $n \in S_i, n > t$ and $y_n \in K_i$ such that

$$\Phi(w_n, y_n) \neq 0.$$

We apply Φ' to (z_t, y_n) and get

$$\Phi'(z_t, y_n) \equiv \sum_{t\leq j\leq n} q_{m_j} (q_{m_t}^{-1} \Phi(w_j, y_n) \not\equiv 0 \mod q_{m_{n+1}} q_{n_t}^{-1}$$

hence $\Phi(z_t, y_n) \neq 0$ and $y_n \in K_i$. This is equivalent to say that $\ker E_i \not\subseteq \ker \Phi(z_t, \ )$ and Definition 2.1 (iv) follows. Hence $(\Phi', \mathfrak{F}_0, \mathfrak{F}_1) \in \mathfrak{F}$ with $\operatorname{Dom} \Phi' = F_0' \oplus F_1$.

Next we extend Φ' under the name Φ'. Let $F_1' = F_1 \oplus fR$ be a free rank-1 extension. We want $\Phi' : F_0' \oplus F_1' \longrightarrow R$ and must define $\Phi'(\ , f) : F_0' \longrightarrow R$. Put $\Phi'(\ , f) \restriction T_n = \epsilon_n \pi_n^* \restriction T_n$ if $n \in S_0$ and $\Phi'(\ , f) \restriction T_n = 0$ otherwise. By linear extension and $F_0 = \bigoplus_{n\in\omega} T_n$ the map $\Phi'(\ , f) : F_0 \longrightarrow R$ is well-defined. It extends further by continuity to

$$\Phi' : F_0' \oplus F_1' \longrightarrow \widehat{R}.$$

Again we must show that $\operatorname{Im}\Phi' \subseteq R$. Note that $\Phi'(w_n, f) = w_n \epsilon_n \pi_n^* = 0$ for $n \in S_0$ from (iii), and $\Phi'(w_n, f) = 0$ for $n \in \omega \setminus S_0$ by the above, hence

$$\Phi'(z, f) = \sum_{n\in\omega} q_{m_n} \Phi'(w_n, f) = 0$$

and $\operatorname{Im}\Phi' \subseteq R$ follows.

We also must check condition (iv) of Definition 2.1 for the new element $f \in F_1'$. Recall K_i $(i \in \omega)$ is a list of all kernels $\ker E_i$ for finite sets in $\mathfrak{F}_0$. Also enumerate all T_n's with $n \in S_0$ and $r_n = 1$ as, say $T^i = T_{n_i}$ $(i \in \omega)$. Choosing the ranks of the $T^{i'}s$ large enough we can find (as before) $y_i' \in K_i \cap T^i$ with $y_i'\pi_{n_i}^* \neq 0$. Then

$$y_i' \in \ker E_i \text{ but } \Phi'(y_i', f) = y_i'\pi_{n_i}^* \neq 0,$$

hence $y_i' \notin \ker \Phi(\ , f)$ as desired for (iv) above.

Finally we must get rid of η by showing that there is no extension $\eta'' \supset \eta$ as stated in the Lemma. Otherwise we have $\Phi' \subset \Phi'' : F_0'' \oplus R \longrightarrow R$ and

$$x_0 R \oplus F''\eta'' = F''$$

for $\operatorname{Dom}\eta'' = F'' \subset_* F_0''$ with $z \in \operatorname{Dom}\eta' \subseteq F''$. Hence $r = \Phi''(z\eta'', f) \in R$. On the other hand

$$r = \Phi''(z\eta'', f) = \Phi''(\sum_{j\in\omega} q_{m_j}(w_j\eta'), f) = \sum_{j\in\omega} q_{m_j}\Phi'(w_j\eta', f) = \sum_{j\in S_0} q_{m_j}\epsilon_j(w_j\eta\pi_j^*) \in \widehat{R}\backslash R$$

is a contradiction. The lemma follows. $\square$

3 Construction of reflexive modules assuming CH

Let $\{S_0, \dots, S_5\}$ be a decomposition of the set of all limit ordinals in ω_1 into stationary sets. Using CH we can enumerate sets of cardinality $2^{\aleph_0}$ by any of these stationary sets of size $\aleph_1$.
Let

$$\mathbf{T} = {}^{\omega_1>}2 = \{\eta_\alpha : \alpha \in \omega_1\} \tag{3.1}$$

be an enumeration of the tree with countable branches such that

$$\eta_\alpha \subseteq \eta_\beta \Longrightarrow \alpha < \beta \tag{3.2}$$

We are working in the 'universe' ω_1 and let

$$\operatorname{Map}(\omega_1, R) = \{\varphi_i : i \in S_0\} = \{\varphi_i : i \in S_1\} = \{\varphi_i : i \in S_2\} = \{\varphi_i : i \in S_3\} \tag{3.3}$$

be four lists of all partial functions $\varphi : \delta \longrightarrow R$ for $\delta \in \omega_1$ any limit ordinal with $\aleph_1$ repetitions φ_i for each φ. Similarly let

$$\operatorname{Map}(\omega_1, \omega_1) = \{\mu_i : i \in S_4\} = \{\mu_i : i \in S_5\}$$

be two lists of all partial maps $\mu : X \longrightarrow X \subseteq \omega_1$ with countable domains X given by the prediction Lemma 1.2 for $E \in \{S_4, S_5\}$.

Inductively we want to construct an order preserving continuous map

$$p : {}^{\omega_1 >}2 = \mathbf{T} \longrightarrow \mathfrak{F},\ (\eta \longrightarrow p_\eta)$$

subject to certain conditions $(a), \dots, (d)$ dictated by the proof of the main theorem and stated below. Some preliminary words are in order. Using the enumeration (3.1) we may write

$$p_{\eta_\alpha} = p_\alpha = (\Phi_\alpha, \mathfrak{F}_{0\alpha}, \mathfrak{F}_{1\alpha}) \text{ and } \operatorname{Dom}\Phi_\alpha = F_{0\alpha} \oplus F_{1\alpha} \tag{3.4}$$

If p is order preserving then $(\eta_\alpha \subseteq \eta_\beta \Longrightarrow p_\alpha \subseteq p_\beta)$ and continuity applies. If $\eta \in {}^{\alpha}2$ has length $\lg(\eta) = \alpha$ and $\alpha < \omega_1$ is a limit ordinal, then $\bigcup_{\beta<\alpha} \eta \restriction \beta = \eta$ and $\eta \restriction \beta \subseteq \eta$, hence

$$\eta \in {}^{\alpha}2 \Longrightarrow\ p_\eta = \bigcup_{\beta<\alpha} p_{\eta\restriction\beta} \tag{3.5}$$

If all $p_{\eta\restriction\beta} \in \mathfrak{F}$ then $p_\eta \in \mathfrak{F}$ by Observation 2.7. Hence we have no problem in defining p at limit stages by continuity. It remains to consider inductive steps for p. If $\eta, \eta' \in {}^{\omega_1>}2$ then $\gamma = \operatorname{br}(\eta, \eta')$ denotes the branching point of η, η', this is to say that

(i) $\gamma < \min\{\lg(\eta), \lg(\eta')\}$

(ii) $\eta \restriction \gamma = \eta' \restriction \gamma$

(iii) $\eta(\gamma) \neq \eta'(\gamma)$.

If η is a branch of length α and $i \in \{0,1\}$ then $\eta' = \eta \wedge \{i\}$ is a branch of length $\alpha+1$ with $\eta' \restriction \alpha = \eta$ and $\eta'(\alpha) = i$. Now we continue defining $p : {}^{\omega_1>}2 \longrightarrow \mathfrak{F}$. Generally we require

(a) If $\varphi \in \mathfrak{F}_{i\alpha}$ *then* $\operatorname{Dom}\varphi = F_{i\beta}$ *for some* $\eta_\beta \subseteq \eta_\alpha$.

(b) *The set* $\{\mathrm{Dom}\,\varphi : \varphi \in \mathfrak{F}_{i\alpha}\}$ *is a well ordered set of pure submodule of* $F_{i\alpha}$*.*

(c) *If* $\gamma = \mathrm{br}\,(\eta_\alpha, \eta_\beta)$ *and* $\eta_\alpha \restriction \gamma = \eta_\beta \restriction \gamma = \eta_\epsilon$ *for some* $\epsilon < \omega_1$*, and* $y \in F_{1\beta} \setminus F_{1\epsilon}$ *then* $\Phi^\beta(y,\ ) \restriction F_{0\epsilon} \in \mathfrak{F}_{0\alpha}$*. Dually, if* $y \in F_{0\beta} \setminus F_{1\beta}$ *then* $\Phi^\beta(y,\ ) \restriction F_{1\epsilon} \in \mathfrak{F}_{1\alpha}$*.*

(d) *If* $\nu \in {}^{\gamma}2$ *is a branch of length* γ*,* $i \in \{0, 1\}$ *and* $\eta = \nu \wedge \{i\}$ *then we want to define* p_η *depending on* γ*.*

   (i) *For* $\nu = \emptyset$ *choose* $p_{\{i\}}$ *by Lemma 2.5.*

   (ii) *If* $\gamma \in S_0$ *then we want to enlarge* Φ_ν *to make the evaluation map injective: If* $\varphi_\gamma \restriction F_\varphi$ *for some* $\varphi \in \mathfrak{F}_{0\nu}$ *is a partial R-homomorphism* $F_\varphi \longrightarrow R$ *with* $\mathrm{Dom}\,(\varphi_\gamma \restriction F_\varphi)$ *a pure R-submodule of finite rank of* F_φ*, then we apply Lemma 2.6 such that* $p_\nu \subset p_\eta$ *with* $F_{1\eta} = F_{1\nu} \oplus x_\eta R$*,* $F_{0\eta} = F_{0\nu}$*,* $\Phi_\nu \subset \Phi_\eta$ *and* $\varphi_\gamma \subset \Phi_\eta(\ , x_\eta)$*. If* φ_γ *does not satisfy the requirements, then we choose any* $\Phi_\nu \subset \Phi_\eta$*.*

   (iii) *If* $\gamma \in S_1$ *then we argue as in (ii) but dually. A dual version of Lemma 2.6 provides* $\Phi_\nu \subset \Phi_\eta$ *and* $\varphi_\gamma \subset \Phi_\eta(x_\eta,\ )$ *if* φ_γ *meets the requirements.*

   (iv) *If* $\gamma \in S_2$ *then we want to kill bad dual maps to make the evaluation map surjective. If* $\varphi_\gamma \restriction F_{0\nu}$ *is an R-homomorphism* $F_{0\nu} \longrightarrow R$ *which is essential for* Φ_ν *then we apply the First Killing Lemma 2.10 to find* $p_\gamma \subset p_\eta$ *such that* $F_{0\eta} = \langle F_{0\nu}, y\rangle \subseteq_* \widehat{F}_{\nu 0}$ *for some* $y \in \widehat{F}_{\nu 0}$ *and* $\varphi_\gamma \restriction F_{0\nu}$ *does not extend to* $F_{0\eta} \longrightarrow R$*.*

   (v) *If* $\gamma \in S_3$ *and* $\varphi_\gamma \restriction F_{1\nu}$ *is an essential R-homomorphism for* Φ_ν *then we argue as in (iv) but dually.*

   (vi) *If* $\gamma \in S_4$ *then we want to get rid of potential monomorphisms* η *of the final module* G *with* $G\eta \oplus xR = G$*. If* $\varphi \in \mathfrak{F}_{0\nu}$*,* $\mu = \varphi_\gamma \restriction F_\varphi$ *and* $\mu : F_\varphi \longrightarrow F_\varphi$ *is an R-monomorphism such that* $F_\varphi = x_\varphi R \oplus F_\varphi \mu$*, then we apply the Second Killing Lemma 2.11 to find* $p_\nu \le p_\eta$ *such that* μ *does not extend to a monomorphism* μ' *of an extension* F'_0 *of* $F_{0\eta}$ *with*

$$p_\eta \subseteq p' = (\Phi', \mathfrak{F}'_0, \mathfrak{F}'_1),\ \mathrm{Dom}\,\Phi' = F'_0 \oplus F'_1 \text{ and } F'_0 = x_\varphi R \oplus F'_0\mu'.$$

   (vii) *If* $\gamma \in S_5$*, then we argue dually for some partial monomorphism* μ *with domain and range some* $F_\varphi \subseteq_* F_{1\nu}$*.*

   (viii) *If* $\gamma \in \omega_1$ *is not a limit ordinal, then we are free to choose any 'trivial' extension* $p_\nu \subset p_\eta$*.*

4 Proof of the Main Theorem

Recall from Section 3 that we are given an order preserving continuous map

$$p : \ \mathbf{T} \ \longrightarrow \mathfrak{F}.$$

By continuity we may extend p to

$$p_\eta = (\Phi_\eta, \mathfrak{F}_{0\eta}, \mathfrak{F}_{1\eta}) \text{ for all } \eta \in {}^{\omega_1}2.$$

It is immediate that

$$F_{i\eta} \text{ is } \aleph_1\text{-free and } \Phi_\eta : F_{0\eta} \oplus F_{1\eta} \longrightarrow R \text{ is a bilinear form,} \tag{4.1}$$

where $i \in \{0,1\},\ \eta \in {}^{\omega_1}2$. Moreover, Φ_η preserves purity and is not degenerated in the sense of Definition 2.1. $\Phi_\eta : F_{0\eta} \oplus F_{1\eta} \longrightarrow R$ is our candidate for a reflexive modules, expressed in an unusual way.

To see that Φ_η preserves purity we must show that each pure element $e \in_* F_{0\eta}$ induces $\Phi_\eta(e,\ ) \in_* F^*_{1\eta}$ (and dually). We may restrict to the first case. If $e \in_* F_{0\eta}$ then $e \in_* F_{0\eta\restriction\alpha}$ for any $\alpha \in \omega_1$ large enough, and $\Phi_{\eta\restriction\alpha}(e,\ ) \in_* F^*_{1\eta\restriction\alpha}$ by $(\Phi_{\eta\restriction\alpha}, \mathfrak{F}_{0\eta\restriction\alpha}, \mathfrak{F}_{1\eta\restriction\alpha}) \in \mathfrak{F}$ and Definition 2.1 (ii). It follows that $\bigcup\limits_{\alpha\in\omega_1} \Phi_{\eta\restriction\alpha}(e,\ ) \in_* F^*_{1\eta}$.

To see that Φ_η is not degenerated we consider $0 \neq e \in F_{0\eta}$. Hence $e \in e'R \subseteq_* F_{0\eta\restriction\alpha}$ for a pure element e' and any $\alpha \in \omega_1$ large enough. The partial homomorphism $\varphi : e'R \longrightarrow R$ defined by $e'\varphi = 1$ has a number $i \in S_0$ in the list and $\varphi = \varphi_i$. By construction there is $y \in F_{1\eta\restriction\gamma}$ with

$$\varphi \subset \Phi_{\eta\restriction\gamma}(\ ,y).$$

Hence $0 \neq e\varphi = \Phi_{\eta\restriction\gamma}(e,y) = \Phi_\eta(e,y)$ and Φ_η is not degenerated. $\square$

Definition 4.1 *We will say that* (Φ, F_0, F_1) *with* $\operatorname{Dom}\Phi = F_0 \oplus F_1$ *is fully represented if* $\Phi_\eta(F_{0\eta},\ ) = F^*_{1\eta}$ *and* $\Phi_\eta(\ ,F_{1\eta}) = F^*_{0\eta}$.

We claim that

$$(\Phi_\eta, F_{0\eta}, F_{1\eta}) \text{ is fully represented for almost all } \eta \in {}^{\omega_1}2 \tag{4.2}$$

There are at most $\eta \in W \subseteq {}^{\omega_1}2$ exceptions with $|W| < 2^{\aleph_1}$.

Suppose for contradiction that $|W| = 2^{\aleph_1}$ and

$$\varphi_\eta \in F^*_{1\eta} \setminus \Phi_\eta(F_{0\eta},\ ) \text{ for all } \eta \in W \tag{4.3}$$

By a pigeon hole argument there are $\eta, \eta' \in W$ with $\mathrm{br}\,(\eta, \eta') = \alpha$ and

(a) $\varphi_\eta \restriction F_{1\alpha} = \varphi_{\eta'} \restriction F_{1\alpha}$

(b) $\varphi_\eta : F_{1\eta} \longrightarrow R$, and $\varphi_{\eta'} : F_{1\eta'} \longrightarrow R$ are not represented.

Let $\psi = \varphi_\eta \restriction F_{1\alpha} = \varphi_{\eta'} \restriction F_{1\alpha} : F_{1\alpha} \longrightarrow R$ and recall that there is some $\gamma \in S_3$ with $\psi = \varphi_\gamma$ and $(d)(v)$ of the construction applies. Hence ψ is inessential for Φ_γ. There is a finite set $E \leq F_{0\gamma}$ such that

$$\Phi_\gamma(e, x) = 0 \text{ for all } e \in E,\ x \in \mathrm{Dom}\,\psi = F_{1\alpha} \text{ implies } x\psi = 0.$$

By Observation 2.9 we have some $e \in \langle E\rangle \subseteq F_{0\gamma}$ such that $\psi = \Phi_\gamma(\ ,e) \restriction F_{1\alpha}$. The same argument applies for $\varphi_{\eta'}$ and there are $\gamma' \in S_3$ and $e' \in F_{0\gamma'}$ such that $\gamma < \gamma'$ and $\psi = \Phi_{\gamma'}(\ ,e') \restriction F_{1\alpha}$. Hence $\psi = \Phi_{\gamma'}(\ ,e') \restriction F_\alpha = \Phi_\gamma(\ ,e) \restriction F_\alpha$.

Finally we apply (c) of the construction to get $\psi \in \mathfrak{F}_{0\gamma'}$. Now Definition 2.1 (iv) applies and $\ker\psi \not\subseteq \ker\Phi_{\gamma'}(\ ,e')$ is a contradiction because $\psi = \Phi_{\gamma'}(\ ,e')$. The claim (4.2) follows. $\square$

Note that we did not use (vi) so far. Hence without the Second Killing Lemma 2.11 we are able to derive reflexivity of the modules G_η, which we will do next.

We will use the following notations.

$$\text{Let } I' = \{\eta \in {}^{\omega_1}2 \text{ such that } p_\eta \text{ is not fully represented}\} \text{ and } I = {}^{\omega_1}2 \setminus I'$$

From (4.2) we see that $|I'| < 2^{\aleph_1}$, hence $|I| = 2^{\aleph_1}$. If $\eta \in I$ and $i \in \{0, 1\}$ then we also fix the evaluation map

$$\sigma_{i\eta} : F_{i\eta} \longrightarrow F_{i\eta}^{**}$$

and claim that

$$\sigma_{i\eta} : F_{i\eta} \longrightarrow F_{i\eta}^{**} \text{ is injective for all } \eta \in I,\ i \in \{0, 1\}. \tag{4.4}$$

Proof. We consider $\sigma = \sigma_{0\eta}$ and apply that Φ_η is not degenerated. If $0 \neq x \in F_{0\eta}$ there is $y \in F_{1\eta}$ such that $\Phi_\eta(x, y) \neq 0$. Hence $\varphi := \Phi_\eta(\ ,y) \in F_{0\eta}^*$ and $x\varphi = \Phi_\eta(x, y) \neq 0$, thus $x\sigma \neq 0$ and σ is injective. The case $\sigma_{1\eta}$ is similar. $\square$

Next we show that

$$\sigma_{i\eta} : F_{i\eta} \longrightarrow F_{i\eta}^{**} \text{ is surjective for all } \eta \in I,\ i \in \{0, 1\}. \tag{4.5}$$

Proof. First note that

$$\Phi_\eta^\bullet : F_{0\eta} \longrightarrow F_{1\eta}^* \ \ (x \longrightarrow \Phi_\eta(x,\ )) \text{ is bijective} \tag{4.6}$$

and

$$^\bullet\Phi_\eta : F_{1\eta} \longrightarrow F^*_{0\eta}(y \longrightarrow \Phi_\eta\ (\ ,y)) \quad \text{is bijective} \tag{4.7}$$

because Φ_η is not degenerated and $\eta \in I$. Hence we can identify $F_{1\eta}$ and $F^*_{0\eta}$ by $^\bullet\Phi_\eta$ and $F^*_{0\eta} = \mathrm{Im}\,(^\bullet\Phi_\eta) = \Phi_\eta(\ , F_{1\eta})$. Moreover

$$F^{**}_{0\eta} = (F^*_{0\eta})^* = (F_{1\eta})^* = \mathrm{Im}\,\Phi^\bullet_\eta = \Phi_\eta\ (F_{0\eta},\ )$$

and for any $\varphi \in F^{**}_{0\eta}$ we find $f \in F_{0\eta}$ with $\varphi = \Phi_\eta(f,\ )$. We consider the case $\sigma = \sigma_{0\eta}$, and get

$$\Phi_\eta(\ ,x)\sigma(f) = \Phi_\eta(f,x) = \Phi_\eta(\ ,x)\Phi_\eta(f,\ )$$

for all $x \in F_{1\eta}$ and $\Phi_\eta(\ ,x)$ runs through all of $F^*_{0\eta}$. We derive $\sigma(f) = \Phi_\eta(f,\ ) = \varphi$ and σ is surjective. The case $\sigma_{1\eta}$ is similar. $\square$

We have an immediate corollary from (4.4) and (4.5).

Corollary 4.2 *If $\eta \in I$ and $i \in \{0,1\}$ then $F_{i\eta}$ is a reflexive R-module. Moreover I is a subset of ${}^{\omega_1}2$ of cardinality $2^{\aleph_1}$.*

For the **Proof of the Main Theorem 1.3** we finally must show that

$$F_{i\eta} \not\cong R \oplus F_{i\eta} \text{ for any } \eta \in I,\ i \in \{0,1\}. \tag{4.8}$$

We consider $F_{i\eta}$ with some monomorphism $\xi : F_{i\eta} \longrightarrow F_{i\eta}$ such that $F_{i\eta} = F_{i\eta}\xi \oplus xR$ for any $\eta \in I, i \in \{0,1\}$. By a back and forth argument there is an $\alpha < \omega_1$ such that $F_{i\eta\restriction\alpha} = F_{i\eta\restriction\alpha}\xi \oplus xR$ and we take $\psi = \xi \restriction F_{i\eta\restriction\alpha}$ into consideration. There is some $\gamma \in S_4$ such that $\psi = \varphi_\gamma$ and φ_γ is discarded by the construction. Hence ξ does not exist. $\square$

We would like to add a modification of our main result which can be shown using one more stationary subset S_6 after introducing inessential endomorphisms for our category of reflexive modules. Let $\mathrm{Fin}\,(G)$ be the ideal of all endomorphisms

$$\{\sigma \in \mathrm{End}\,(G) :\ G\sigma \text{ has finite rank}\}$$

for some torsion-free R-module G. Then we can find a 'Killing-Lemma' in Dugas, Göbel [4], see also [2], which 'takes care' of all endomorphisms which are not in $\mathrm{Fin}\,(F_{i\eta})$ with i, η as above. Hence we can strengthen our Main Theorem 1.3 and get with slight modification from known results the following

Corollary 4.3 *(ZFC + CH) Let R is a countable domain but not a field and A be a countable R-algebra with free additive structure A_R. Then there is a family of $2^{\aleph_1}$ pair-wise non-isomorphic reflexive R-modules G of cardinality $\aleph_1$ such that $G \not\cong R \oplus G$ and* $\mathrm{End}\,(G) = A \oplus \mathrm{Fin}\,(G)$ *a split extension.*

In particular $\mathrm{End}\,(G)/\mathrm{Fin}\,(G) \cong A$ and if A has only trivial idempotents like $A = R$ then G in the Corollary 4.3 is reflexive, essentially indecomposable of size $\aleph_1$ and does not decompose into $G \cong R \oplus G$.

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